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Dual polar graphs, a nil-DAHA of rank one, and non-symmetric dual q-Krawtchouk polynomials

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Abstract. Let Γ be a dual polar graph with diameter $D \geq 3$. From every pair of a vertex of Γ and a maximal clique containing it, we construct a $2D$-dimensional irreducible module for a nil-DAHA of type $(C_1^\vee, C_1)$. Using this module, we define non-symmetric dual q-Krawtchouk polynomials and describe their orthogonality relations.

Keywords: Dual polar graph, Terwilliger algebra, nil-DAHA of rank one, dual q-Krawtchouk polynomial

1 Introduction

Q-polynomial distance-regular graphs (DRGs) are viewed as finite analogues of compact symmetric spaces of rank one, and have been extensively studied; cf. [1, 2, 7]. By a famous theorem of Leonard [11], [1, §3.5], the duality property of Q-polynomial DRGs characterizes the terminating branch of the Askey scheme [8] of (basic) hypergeometric orthogonal polynomials, at the top (i.e., ${}_4\phi_3$) of which are the q-Racah polynomials. A central tool in studying such a graph is the *Terwilliger algebra* $T = T(x)$ [14], which is a non-commutative semisimple matrix $\mathbb{C}$-algebra attached to every vertex x of the graph.

The *double affine Hecke algebras* (DAHAs) for reduced affine root systems were introduced by Cherednik [3] in his proof of Macdonald's constant term conjecture for Macdonald polynomials. Sahi [12] extended the definition of DAHAs to the non-reduced affine root systems of type $(C_n^\vee, C_n)$, and proved the duality conjecture for the Koornwinder polynomials, which are the Macdonald polynomials attached to the affine root systems of type $(C_n^\vee, C_n)$. For $n = 1$, these polynomials are the Askey–Wilson polynomials which are of ${}_4\phi_3$, and the q-Racah polynomials are a discretization of the Askey–Wilson polynomials.

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Recently, the first author [9] found a link between the theories of Q-polynomial DRGs and the DAHAs. Namely, he considered a Q-polynomial DRG Γ corresponding to q-Racah polynomials. He further assumed that Γ possesses a clique C with maximal possible size (called a *Delsarte clique*), and defined a semisimple matrix $\mathbb{C}$-algebra $\mathbf{T} = \mathbf{T}(x, C)$ attached to C *and* a vertex $x \in C$, which contains $T(x)$ as a subalgebra. Then he showed that the so-called *primary* $\mathbf{T}$-module has the structure of an irreducible module for the DAHA of type $(C_1^\vee, C_1)$, and studied how the two module structures are related. In the subsequent paper [10], he captured in this context what should be called the *non-symmetric q-Racah polynomials*, which are the finite counterpart of the non-symmetric Askey–Wilson polynomials discussed by Sahi [12], and succeeded in describing their orthogonality relations explicitly.

A big goal in this project is to establish a "non-symmetric version" of Leonard's theorem mentioned above. As the next attempt towards this goal, we discuss the *dual polar graphs* in this extended abstract, and specialize the above situation to this case. The dual polar graphs are a classical family of Q-polynomial DRGs, and correspond to dual q-Krawtchouk polynomials which are of ${}_3\phi_2$. In particular, we will obtain the *non-symmetric* dual q-Krawtchouk polynomials and describe their orthogonality relations; cf. Theorem 7.6. There are multiple motivations for the research presented here. First, for the q-Racah case, there is indeed no known example of a Q-polynomial DRG having such a maximal clique, so that the theory developed in [9, 10] remains at the algebraic/parametric level, whereas we will deal with concrete combinatorial examples in this extended abstract. Second, there are of course other candidates of examples, such as the Grassmann graphs corresponding to the dual q-Hahn polynomials which lie in between the q-Racah and the dual q-Krawtchouk polynomials, but we decided to focus on the dual polar graphs, mainly because they exhibit quite a strong regularity of being *regular near polygons*, so that the computations become far simpler than those in [9, 10]. Though many of our results can also be obtained in principle by taking appropriate limits of the (much involved) results in [9, 10], this fact motivates us to work out the details for this case rather independently of [9, 10]. Third, we will encounter a *nil-DAHA* of type $(C_1^\vee, C_1)$, which is obtained by specializing some of the defining relations of the DAHA of type $(C_1^\vee, C_1)$. The nil-DAHAs were introduced and discussed recently by Cherednik and Orr [4, 5, 6], and our results demonstrate the fundamental importance of the concept in the theory of Q-polynomial DRGs; cf. Theorems 5.7 and 5.8.

Throughout this extended abstract, we use the following notation. For a given nonempty finite set X, let $\mathrm{Mat}_X(\mathbb{C})$ be the $\mathbb{C}$-algebra consisting of the complex square matrices indexed by X. Let $V = V_X$ be the $\mathbb{C}$-vector space consisting of the complex column vectors indexed by X. We endow V with the standard Hermitian inner product $\langle u, v \rangle = u^t \overline{v}$ for $u, v \in V$. For every $y \in X$, let $\hat{y}$ be the vector in V with a 1 in the y-coordinate and 0 elsewhere. For a subset $Y \subseteq X$, let $\hat{Y} = \sum_{y \in Y} \hat{y} \in V$. A Laurent polynomial $f(\eta) \in \mathbb{C}[\eta, \eta^{-1}]$ in the variable η is said to be *symmetric* if $f(\eta) = f(\eta^{-1})$, and *non-symmetric* otherwise. Note that the symmetric Laurent polynomials are precisely the polynomials in $\xi := \eta + \eta^{-1}$. Let q be a prime power. For $a \in \mathbb{C}$ and an integer $n \geq 0$, let

$$(a;q)_n = (1-a)(1-aq)\dots(1-aq^{n-1}), \qquad \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ 1 \end{bmatrix}_q = \frac{q^n-1}{q-1}.$$

2 Preliminaries: Distance-regular graphs

Let Γ be a finite simple connected graph with vertex set X and diameter D. For $x \in X$, let $\Gamma_i(x) = \{y \in X : \partial(x,y) = i\}$ for $0 \leq i \leq D$, where ∂ denotes the shortest path-length distance. We abbreviate $\Gamma(x) := \Gamma_1(x)$. We call Γ *distance-regular* if there are integers a_i, b_i, c_i $(0 \leq i \leq D)$, called the *intersection numbers* of Γ, such that

$$a_i = |\Gamma_i(x) \cap \Gamma(y)|, \qquad b_i = |\Gamma_{i+1}(x) \cap \Gamma(y)|, \qquad c_i = |\Gamma_{i-1}(x) \cap \Gamma(y)|$$

for every pair of vertices $x, y \in X$ with $\partial(x,y) = i$, where $\Gamma_{-1}(x) = \Gamma_{D+1}(x) := \emptyset$.

Assume that Γ is distance-regular. The i^{th} *distance matrix* of Γ is the 0-1 matrix $A_i \in \mathrm{Mat}_X(\mathbb{C})$ such that $(A_i)_{xy} = 1$ if and only if $\partial(x,y) = i$. The *Bose-Mesner algebra* of Γ is the semisimple subalgebra M of $\mathrm{Mat}_X(\mathbb{C})$ generated by the A_i. Observe that

$$A_1 A_i = b_{i-1}A_{i-1} + a_i A_i + c_{i+1}A_{i+1} \qquad (0 \leq i \leq D),$$

from which it follows that for $0 \leq i \leq D$, there is a polynomial $f_i \in \mathbb{C}[\xi]$ with $\deg f_i = i$ such that $f_i(A) = A_i$. In particular, the adjacency matrix $A := A_1$ of Γ generates M.

Since A is real symmetric and generates M, it has $D+1$ mutually distinct real eigenvalues $\theta_0, \theta_1, \cdots, \theta_D$, which we call the *eigenvalues of* Γ. We will always set $\theta_0 := b_0$, the valency (or degree) of Γ. For $0 \leq i \leq D$, let $E_i \in \mathrm{Mat}_X(\mathbb{C})$ be the orthogonal projection onto the eigenspace of θ_i. Then we have $A = \sum_{i=0}^{D} \theta_i E_i$, so that the E_i form a basis for M. Note that M is also closed under entrywise multiplication, denoted $\circ$. We say that Γ is *Q-polynomial* with respect to the ordering $\{E_i\}_{i=0}^{D}$ (or $\{\theta_i\}_{i=0}^{D}$) if there are scalars a_i^*, b_i^*, c_i^* $(0 \leq i \leq D)$ such that $b_D^* = c_0^* = 0$, $b_{i-1}^* c_i^* \neq 0$ $(1 \leq i \leq D)$, and

$$E_1 \circ E_i = \frac{1}{|X|}(b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1}) \qquad (0 \leq i \leq D),$$

where we set $b_{-1}^* E_{-1} = c_{D+1}^* E_{D+1} := 0$. If this is the case, then for $0 \leq i \leq D$, there is a polynomial $f_i^* \in \mathbb{C}[\xi]$ with $\deg f_i^* = i$ such that $f_i^*(E_1) = E_i$, where the multiplication is under $\circ$. In particular, if we write $E_1 = |X|^{-1} \sum_{i=0}^{D} \theta_i^* A_i$, then the θ_i^* are (real and) mutually distinct. Note also that $\theta_0^* = \operatorname{trace} E_1 = \operatorname{rank} E_1$.

Assume that Γ is Q-polynomial with respect to the ordering $\{E_i\}_{i=0}^{D}$. Fix a vertex $x \in X$. The *dual adjacency matrix* of Γ with respect to x is the diagonal matrix $A^* = A^*(x) \in \mathrm{Mat}_X(\mathbb{C})$ defined by $(A^*)_{yy} = |X|(E_1)_{xy}$ for $y \in X$. Note that the θ_i^* are the eigenvalues of A^*, which we call the *dual eigenvalues* of Γ. The *Terwilliger* (or *subconstituent*) *algebra* $T = T(x)$ with respect to x is the semisimple subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by A, A^* [14]. The subspace $M\hat{x} = \sum_{i=0}^{D} \mathbb{C}A_i\hat{x} = \sum_{i=0}^{D} \mathbb{C}E_i\hat{x}$ of V turns out to be an irreducible T-module with dimension $D+1$, called the *primary* T-module.

For more detailed information, see [1, 2, 7].

3 Dual polar graphs

In this section, we discuss dual polar graphs. We begin by summarizing some results from [2, §9.4] that we need. Let D be a positive integer. Let $\mathbb{V}$ denote one of the following spaces over the finite field $\mathbb{F}_q$ equipped with a non-degenerate form:

Name	$\dim \mathbb{V}$	Form	e
$[C_D(q)]$	$2D$	alternating	1
$[B_D(q)]$	$2D+1$	quadratic	1
$[D_D(q)]$	$2D$	quadratic (maximal Witt index D)	0
$[^2D_{D+1}(q)]$	$2D+2$	quadratic (non-maximal Witt index D)	2
$[^2A_{2D}(r)]$	$2D+1$	Hermitian ($q = r^2$)	$\frac{3}{2}$
$[^2A_{2D-1}(r)]$	$2D$	Hermitian ($q = r^2$)	$\frac{1}{2}$

We note that maximal (totally) isotropic subspaces have dimension D. Let X be the set of all maximal isotropic subspaces of $\mathbb{V}$. The *dual polar graph* (on $\mathbb{V}$) has vertex set X, where two vertices x, y are adjacent whenever $\dim(x \cap y) = D - 1$. This graph is distance-regular and has diameter D. For the rest of this extended abstract, we shall assume that Γ is a dual polar graph with diameter $D \geq 3$.

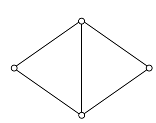
The intersection numbers and the eigenvalues of Γ are given by

$$a_i = (q^e - 1)\begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad b_i = q^{i+e}\begin{bmatrix} D-i \\ 1 \end{bmatrix}, \qquad c_i = \begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad \theta_i = q^e\begin{bmatrix} D-i \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}$$

for $0 \leq i \leq D$. Moreover, Γ is Q-polynomial with respect to the ordering $\{\theta_i\}_{i=0}^{D}$. The dual eigenvalues of Γ are given by

$$\theta_i^* = \frac{q(1+q^{D+e-2})}{1-q} + \frac{q(1+q^{D+e-2})(1+q^{D+e-1})}{(q-1)(1+q^{e-1})}q^{-i}$$

for $0 \leq i \leq D$; cf. [15, Lemma 16.5]. The dual polar graph Γ is an example of a *regular near polygon* (cf. [2, §6.4]), which means that Γ does not have



(i.e., $K_{1,1,2}$) as an induced subgraph, and that for every $x \in X$ and a maximal clique C, there is a unique $y \in C$ nearest to x, provided that $\partial(x, C) < D$. Note that the former condition implies that every edge lies in a unique maximal clique.

Let C be a maximal clique of Γ. For $0 \leq i \leq D-1$, define $C_i = \{y \in X : \partial(y, C) = i\}$, called the i^{th} *distance neighbor* of C. By [9, Corollary 4.8] (cf. [2, §11.1]), $\{C_i\}_{i=0}^{D-1}$ is an equitable partition of X, that is, there are integers $\widetilde{a}_i, \widetilde{b}_i, \widetilde{c}_i$ $(0 \leq i \leq D-1)$ such that

$$\widetilde{a}_i = |C_i \cap \Gamma(y)|, \qquad \widetilde{b}_i = |C_{i+1} \cap \Gamma(y)|, \qquad \widetilde{c}_i = |C_{i-1} \cap \Gamma(y)|$$

for every $y \in C_i$, where $C_{-1} = C_D := \emptyset$. It follows that

$$\widetilde{a}_i = q^e \begin{bmatrix} i+1 \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix}, \qquad \widetilde{b}_i = q^{i+1+e} \begin{bmatrix} D-i-1 \\ 1 \end{bmatrix}, \qquad \widetilde{c}_i = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

for $0 \leq i \leq D-1$.

We now recall the Terwilliger algebra of Γ *with respect to* C; cf. [9, §4], [13]. We call the diagonal matrix $\widetilde{A}^* = \widetilde{A}^*(C) := |C|^{-1} \sum_{y \in C} A^*(y)$ the *dual adjacency matrix* of Γ with respect to C. It follows that $\widetilde{A}^*$ has D mutually distinct real eigenvalues

$$\widetilde{\theta}_i^* = \frac{q(1+q^{D+e-2})}{1-q} + \frac{q(1+q^{D+e-2})(1+q^{D+e-1})}{(q-1)(1+q^e)} q^{-i}$$

for $0 \leq i \leq D-1$. The *Terwilliger algebra* $\widetilde{T} = \widetilde{T}(C)$ with respect to C is the semisimple subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $A, \widetilde{A}^*$. The subspace $M\hat{C} = \sum_{i=0}^{D-1} \mathbb{C}\hat{C}_i$ of V is an irreducible $\widetilde{T}$-module with dimension D, called the *primary* $\widetilde{T}$-module.

4 The primary T-module W

We continue to discuss the dual polar graph Γ. For the rest of this extended abstract, we fix a vertex $x \in X$ and a maximal clique C containing x. Recall $T = T(x)$ and $\widetilde{T} = \widetilde{T}(C)$.

Definition 4.1 ([9, Definition 5.20])**.** The *generalized Terwilliger algebra* of Γ with respect to x, C is the semisimple subalgebra $\mathbf{T} = \mathbf{T}(x, C)$ of $\mathrm{Mat}_X(\mathbb{C})$ generated by $T, \widetilde{T}$.

Note that $A, A^*, \widetilde{A}^*$ generate $\mathbf{T}$ by definition. We now construct an irreducible $\mathbf{T}$-module. Recall the equitable partition $\{C_i\}_{i=0}^{D-1}$ of X. For $0 \leq i \leq D-1$, let

$$C_i^- = \Gamma_i(x) \cap C_i, \qquad C_i^+ = \Gamma_{i+1}(x) \cap C_i.$$

Then, it follows that

$$|C_i^-| = q^{ie} \prod_{j=1}^{i} \frac{q^D - q^j}{q^j - 1}, \qquad |C_i^+| = q^{(i+1)e} \prod_{j=1}^{i} \frac{q^D - q^j}{q^j - 1} \tag{4.1}$$

for $0 \leq i \leq D-1$. In particular, the $C_i^{\pm}$ are nonempty. Moreover, it turns out that $\{C_i^{\pm}\}_{i=0}^{D-1}$ is again an equitable partition of X. Let $\mathbf{W}$ be the subspace of V spanned by the $\hat{C}_i^{\pm}$. Consider the following ordered orthogonal basis for $\mathbf{W}$:

$$\mathcal{C} = \{\hat{C}_0^-, \hat{C}_0^+, \hat{C}_1^-, \hat{C}_1^+, \ldots, \hat{C}_{D-1}^-, \hat{C}_{D-1}^+\}. \tag{4.2}$$

Lemma 4.2. *For $0 \leq i \leq D-1$, we have*

$$\begin{aligned} A.\hat{C}_i^- &= \frac{q^{D+e}-q^{i+e}}{q-1}\hat{C}_{i-1}^- + (q^e-1)\frac{q^i-1}{q-1}\hat{C}_i^- + q^i\hat{C}_i^+ + \frac{q^{i+1}-1}{q-1}\hat{C}_{i+1}^-, \\ A.\hat{C}_i^+ &= \frac{q^{D+e}-q^{i+e}}{q-1}\hat{C}_{i-1}^+ + q^{e+i}\hat{C}_i^- + (q^e-1)\frac{q^{i+1}-1}{q-1}\hat{C}_i^+ + \frac{q^{i+1}-1}{q-1}\hat{C}_{i+1}^+, \end{aligned}$$

where $\hat{C}_{-1}^- = \hat{C}_{-1}^+ = \hat{C}_D^- = \hat{C}_D^+ := 0$.

Lemma 4.3. *For $0 \leq i \leq D-1$, we have*

$$\begin{aligned} A^*.\hat{C}_i^- &= (\alpha + \beta q^{-i})\hat{C}_i^-, & A^*.\hat{C}_i^+ &= (\alpha + \beta q^{-i-1})\hat{C}_i^+, \\ \widetilde{A}^*.\hat{C}_i^- &= (\alpha + \widetilde{\beta} q^{-i})\hat{C}_i^-, & \widetilde{A}^*.\hat{C}_i^+ &= (\alpha + \widetilde{\beta} q^{-i})\hat{C}_i^+, \end{aligned}$$

where

$$\alpha = \frac{q(1+q^{D+e-2})}{1-q},$$

and

$$\beta = \frac{q(1+q^{D+e-2})(1+q^{D+e-1})}{(q-1)(1+q^{e-1})}, \qquad \widetilde{\beta} = \frac{q(1+q^{D+e-2})(1+q^{D+e-1})}{(q-1)(1+q^{e})}.$$

Proposition 4.4. *The subspace $\mathbf{W}$ is an irreducible $\mathbf{T}$-module.*

We call $\mathbf{W}$ the *primary* $\mathbf{T}$-module. Note that the primary T-module $M\hat{x}$ is a subspace of $\mathbf{W}$. Indeed, we have

$$\hat{x} = \hat{C}_0^-, \qquad A_i\hat{x} = \hat{C}_i^- + \hat{C}_{i-1}^+ \quad (1 \leq i \leq D-1), \qquad A_D\hat{x} = \hat{C}_{D-1}^+. \tag{4.3}$$

Let $M\hat{x}^{\perp}$ be the orthogonal complement of $M\hat{x}$ in $\mathbf{W}$. Then it turns out that $M\hat{x}^{\perp}$ is also an irreducible T-module. For $0 \leq i \leq D-2$, let

$$v_i^{\perp} = (q^{D-i-1}-1)\hat{C}_i^+ + (q^{-i-1}-1)\hat{C}_{i+1}^-. \tag{4.4}$$

It follows from (4.1) and (4.3) that the $v_i^{\perp}$ form a basis for $M\hat{x}^{\perp}$. It can also be shown that the vectors $E_iv_0^{\perp}$ $(1 \leq i \leq D-1)$ form a basis for $M\hat{x}^{\perp}$.

5 A nil-DAHA of type $(C_1^\vee, C_1)$

For type $(C_1^\vee, C_1)$, there is some flexibility in the definition of a nil-DAHA. It will turn out that the following specialization is the one which is well-suited to our situation:

Definition 5.1. Let $r_0, r_1 \in \mathbb{C}$ be nonzero scalars. Let $\overline{H} = \overline{H}(r_0, r_1)$ be the $\mathbb{C}$-algebra defined by generators t_0, u_0, t_1, u_1 and relations (i) $(t_n - r_n)(t_n - r_n^{-1}) = 0$ for $n \in \{0,1\}$; (ii) $u_0^2 = u_0$; (iii) $u_1^2 = 0$; (iv) $(u_0t_0)(t_1u_1) = 0 = (t_1u_1)(u_0t_0)$. We call $\overline{H}$ a *nil-DAHA of type* $(C_1^\vee, C_1)$.

By Definition 5.1(i) we have $t_n((r_n + r_n^{-1}) - t_n) = 1 = ((r_n + r_n^{-1}) - t_n)t_n$ for $n \in \{0,1\}$, from which it follows that t_0, t_1 are invertible, and that $t_0 + t_0^{-1}, t_1 + t_1^{-1}$ are central.

For the rest of the extended abstract, we fix $a \in \mathbb{C}$ such that $a^2 = -1/q^{D+e}$, and set

$$r_0 = q^{-D/2}, \qquad r_1 = aq^{D/2}.$$

We now define a $2D$-dimensional representation of $\overline{H}$.

Definition 5.2. (i) For $1 \le i \le D-1$, let

$$t_0(i) = \begin{pmatrix} q^{-D/2}(q^D - q^i + 1) & q^{D/2}(q^{i-D} - 1) \\ q^{-D/2}(1 - q^i) & q^{-D/2+i} \end{pmatrix}, \qquad u_0(i) = \begin{pmatrix} 1 & q^{D-i} - 1 \\ 0 & 0 \end{pmatrix}.$$

Let $t_0(0) = \left(q^{-D/2}\right)$, $t_0(D) = \left(q^{-D/2}\right)$, $u_0(0) = (0)$, and $u_0(D) = (1)$.

(ii) For $0 \le i \le D-1$, let

$$t_1(i) = \begin{pmatrix} aq^{D/2} + a^{-1}q^{-D/2} & -a^{-1}q^{-D/2} \\ aq^{D/2} & 0 \end{pmatrix}, \qquad u_1(i) = \begin{pmatrix} 0 & 0 \\ -aq^{D/2-i} & 0 \end{pmatrix}.$$

Referring to Definition 5.2, consider the following $2D \times 2D$ block diagonal matrices:

$$\begin{aligned}
\mathcal{T}_0 &= \mathrm{blockdiag}\left(t_0(0), t_0(1), \ldots, t_0(D-1), t_0(D)\right), \\
\mathcal{U}_0 &= \mathrm{blockdiag}\left(u_0(0), u_0(1), \ldots, u_0(D-1), u_0(D)\right), \\
\mathcal{T}_1 &= \mathrm{blockdiag}\left(t_1(0), t_1(1), \ldots, t_1(D-1)\right), \\
\mathcal{U}_1 &= \mathrm{blockdiag}\left(u_1(0), u_1(1), \ldots, u_1(D-1)\right).
\end{aligned}$$

Proposition 5.3. *$\mathcal{T}_0, \mathcal{U}_0, \mathcal{T}_1, \mathcal{U}_1$ satisfy the relations (i)–(iv) in Definition 5.1, and hence define a representation of $\overline{H}$.*

Corollary 5.4. *The primary $\mathbf{T}$-module $\mathbf{W}$ has a module structure for the algebra $\overline{H}$ such that, for $n \in \{0,1\}$, $\mathcal{T}_n$ (respectively $\mathcal{U}_n$) is the matrix representing the action of t_n (respectively u_n) with respect to the ordered basis $\mathcal{C}$ from (4.2).*

We note that $\mathcal{U}_0\mathcal{T}_0$ and $\mathcal{T}_1\mathcal{U}_1$ are diagonal matrices as follows:

$$\mathcal{U}_0\mathcal{T}_0 = \operatorname{diag}\left(0, q^{\frac{D}{2}-1}, 0, q^{\frac{D}{2}-2}, 0, q^{\frac{D}{2}-3}, 0, \dots, q^{-\frac{D}{2}+1}, 0, q^{-\frac{D}{2}}\right),$$
$$\mathcal{T}_1\mathcal{U}_1 = \operatorname{diag}\left(1, 0, q^{-1}, 0, q^{-2}, 0, q^{-3}, 0, \dots, q^{-D+1}, 0\right).$$

By Corollary 5.4, $\mathbf{W}$ is now a module for both $\mathbf{T}$ and $\overline{H}$. We next discuss how the two module structures are related. Let $\mathbf{Y} = t_0t_1$, $\mathbf{X}_0 = u_0t_0$, $\mathbf{X}_1 = t_1u_1$, and let

$$\mathbf{A} = \mathbf{Y} + \mathbf{Y}^{-1}, \qquad \mathbf{B} = q^{-D/2}\mathbf{X}_0 + \mathbf{X}_1, \qquad \widetilde{\mathbf{B}} = q^{-\frac{D}{2}+1}\mathbf{X}_0 + \mathbf{X}_1.$$

Lemma 5.5. *For $0 \le i \le D-1$, the actions of $\mathbf{A}$ on $\hat{C}_i^-, \hat{C}_i^+$ are given respectively as linear combinations with the following terms and coefficients.*

$\hat{C}_i^-$:

term	coefficient
$\hat{C}_{i-1}^-$	$a^{-1}(1-q^{i-D})$
$\hat{C}_{i-1}^+$	0
$\hat{C}_{i}^-$	$(aq^D + a^{-1})q^{i-D}$
$\hat{C}_{i}^+$	$aq^i(1-q)$
$\hat{C}_{i+1}^-$	$a(1-q^{i+1})$

$\hat{C}_i^+$:

term	coefficient
$\hat{C}_{i-1}^+$	$a^{-1}(1-q^{i-D})$
$\hat{C}_{i}^-$	$a^{-1}(q-1)q^{i-D}$
$\hat{C}_{i}^+$	$(aq^D + a^{-1})q^{i-D+1}$
$\hat{C}_{i+1}^-$	0
$\hat{C}_{i+1}^+$	$a(1-q^{i+1})$

Lemma 5.6. *For $0 \le i \le D-1$, the actions of $\mathbf{B}$ and $\widetilde{\mathbf{B}}$ on $\hat{C}_i^-, \hat{C}_i^+$ are as follows.*

$$\mathbf{B}.\hat{C}_i^- = q^{-i}\hat{C}_i^-, \qquad \mathbf{B}.\hat{C}_i^+ = q^{-i-1}\hat{C}_i^+,$$
$$\widetilde{\mathbf{B}}.\hat{C}_i^- = q^{-i}\hat{C}_i^-, \qquad \widetilde{\mathbf{B}}.\hat{C}_i^+ = q^{-i}\hat{C}_i^+.$$

Recall the generators A, A^*, $\widetilde{A}^*$ of $\mathbf{T}$. We now present our first main result.

Theorem 5.7. *On $\mathbf{W}$, we have*

$$A = \frac{aq^{D+e}}{q-1}\mathbf{A} + \frac{1-q^e}{q-1}, \qquad A^* = \beta\mathbf{B} + \alpha, \qquad \widetilde{A}^* = \widetilde{\beta}\widetilde{\mathbf{B}} + \alpha,$$

where $\alpha, \beta, \widetilde{\beta}$ are from Lemma 4.3.

Thus, the actions of $A, A^*, \widetilde{A}^*$ on $\mathbf{W}$ coincide with those of $\mathbf{A}, \mathbf{B}, \widetilde{\mathbf{B}}$, respectively, up to affine transformation.

Let π (respectively $\widetilde{\pi}$) denote the orthogonal projection from $\mathbf{W}$ onto $M\hat{x}$ (respectively $M\hat{C}$). The following result illustrates (to some extent) how we arrived at the $\overline{H}$-module structure on $\mathbf{W}$ given above:

Theorem 5.8. *On $\mathbf{W}$, we have*

$$\pi = \frac{t_0 - q^{D/2}}{q^{-D/2} - q^{D/2}}, \qquad \widetilde{\pi} = \frac{t_1 - a^{-1}q^{-D/2}}{aq^{D/2} - a^{-1}q^{-D/2}}.$$

6 Non-symmetric dual q-Krawtchouk polynomials

In this section, we define a certain finite sequence of Laurent polynomials in one variable η, and show how these Laurent polynomials play a role in the $\overline{H}$-module $\mathbf{W}$. We begin by recalling the (monic) dual q-Krawtchouk polynomials

$$K_i(\xi) = K_i(\xi; a, D; q) = \frac{(q^{-D};q)_i}{a^i} {}_3\phi_2 \left(\begin{matrix} q^{-i},\ a\eta,\ a\eta^{-1} \\ 0,\ q^{-D} \end{matrix} \middle| q, q \right) \qquad (0 \leq i \leq D),$$

where $\xi = \eta + \eta^{-1}$. Recall the basis $\{A_i \hat{x}\}_{i=0}^{D}$ for $M\hat{x}$; cf. (4.3). Then it follows that

$$K_i(\mathbf{Y} + \mathbf{Y}^{-1}).\hat{x} = a^i (q;q)_i \, A_i \hat{x} \qquad (0 \leq i \leq D). \tag{6.1}$$

Consider another set of dual q-Krawtchouk polynomials

$$\begin{aligned} K_i^{\perp}(\xi) &= K_i(\xi; aq, D-2; q) \\ &= \frac{(q^{-D+2};q)_i}{a^i q^i} {}_3\phi_2 \left(\begin{matrix} q^{-i},\ aq\eta,\ aq\eta^{-1} \\ 0,\ q^{-D+2} \end{matrix} \middle| q, q \right) \qquad (0 \leq i \leq D-2). \end{aligned}$$

Recall the basis $\{v_i^{\perp}\}_{i=0}^{D-2}$ for $M\hat{x}^{\perp}$ from (4.4). Then it follows that

$$K_i^{\perp}(\mathbf{Y} + \mathbf{Y}^{-1}).v_0^{\perp} = a^i q^i (q;q)_i \, v_i^{\perp} \qquad (0 \leq i \leq D-2). \tag{6.2}$$

Define $g \in \mathbb{C}[\eta, \eta^{-1}]$ by

$$g(\eta) = \eta^{-1}(\eta - a)(\eta - a^{-1}q^{-D}).$$

Then we have

$$g(\mathbf{Y}).\hat{x} = aq \, v_0^{\perp}. \tag{6.3}$$

From (4.3), (4.4), (6.1), (6.2), and (6.3) it follows that

Lemma 6.1. *For $1 \leq i \leq D-1$, we have*

$$\hat{C}_{i-1}^{+} = \frac{1}{(1-q^D)a^i(q;q)_{i-1}} \Big(K_i(\mathbf{Y} + \mathbf{Y}^{-1}) - K_{i-1}^{\perp}(\mathbf{Y} + \mathbf{Y}^{-1}) g(\mathbf{Y}) \Big) \hat{x},$$

$$\hat{C}_{i}^{-} = \frac{q^D - q^i}{(q^D - 1)a^i(q;q)_{i}} \left(K_i(\mathbf{Y} + \mathbf{Y}^{-1}) - \frac{1-q^i}{q^D - q^i} K_{i-1}^{\perp}(\mathbf{Y} + \mathbf{Y}^{-1}) g(\mathbf{Y}) \right) \hat{x}.$$

In view of Lemma 6.1, we now make the following definition.

Definition 6.2. For $1 \leq i \leq D-1$, let

$$\ell_{i-1}^{+}(\eta) = \frac{K_i - gK_{i-1}^{\perp}}{(1-q^D)a^i(q;q)_{i-1}}, \qquad \ell_i^{-}(\eta) = \frac{q^D - q^i}{(q^D-1)a^i(q;q)_i} \left(K_i - \frac{1-q^i}{q^D - q^i} gK_{i-1}^{\perp} \right).$$

Moreover, let $\ell_0^{-}(\eta) = 1$ and $\ell_{D-1}^{+}(\eta) = K_D / a^D (q;q)_D$. We call the $\ell_i^{\pm}$ the *non-symmetric dual q-Krawtchouk polynomials*.

By definition, the $\ell_i^{\pm}$ are linearly independent in $\mathbb{C}[\eta, \eta^{-1}]$. Observe that

$$\ell_i^{-}(\mathbf{Y}).\hat{x} = \hat{C}_i^{-}, \qquad \ell_i^{+}(\mathbf{Y}).\hat{x} = \hat{C}_i^{+} \qquad (0 \leq i \leq D-1).$$

7 Orthogonality relations

Let $\mathcal{L}$ be the subspace of $\mathbb{C}[\eta,\eta^{-1}]$ spanned by the Laurent polynomials $\ell_i^{\pm}$. In this section, we define a Hermitian inner product on $\mathcal{L}$ and show that the $\ell_i^{\pm}$ are orthogonal with respect to that inner product. Recall the basis $\{E_i\hat{x}\}_{i=0}^{D}$ (respectively $\{E_i v_0^{\perp}\}_{i=1}^{D-1}$) for $M\hat{x}$ (respectively $M\hat{x}^{\perp}$). Consider the following ordered basis for $\mathbf{W}$:

$$\mathfrak{B} = \{E_0\hat{x}, E_1\hat{x}, E_1v_0^{\perp}, E_2\hat{x}, E_2v_0^{\perp}, \ldots, E_{D-1}\hat{x}, E_{D-1}v_0^{\perp}, E_D\hat{x}\}.$$

Lemma 7.1. *The matrix representing the action of $\mathbf{Y} = t_0t_1$ on $\mathbf{W}$ with respect to $\mathfrak{B}$ is*

$$\mathrm{blockdiag}\Big([a], [\mathbf{Y}(1)], [\mathbf{Y}(2)], \ldots, [\mathbf{Y}(D-1)], [a^{-1}q^{-D}]\Big),$$

where for $1 \leq i \leq D-1$, $[\mathbf{Y}(i)]$ is the 2×2 matrix given by

$$\begin{bmatrix} \frac{a(q^{D-i}-1)(q^e+q^i)+a^{-1}q^{-D}(q^i-1)(q^{D+e-i}+1)}{(q^e+1)(q^D-1)} & \frac{(a-a^{-1}q^{-D})(q^e+q^i)(q^{D-i}-1)(q^i-1)(q^{D+e-i}+1)}{q(q^e+1)(q^D-1)} \\ \frac{q(aq^D-a^{-1})}{(q^D-1)(q^e+1)} & \frac{aq^D(q^{D+e-i}+1)(q^i-1)+a^{-1}(q^{D-i}-1)(q^e+q^i)}{(q^D-1)(q^e+1)} \end{bmatrix}.$$

Corollary 7.2. *The eigenvalues of $\mathbf{Y}$ on $\mathbf{W}$ are*

$$\begin{array}{llllll} a, & aq, & aq^2, & \ldots, & aq^{D-1}, & \\ & a^{-1}q^{-1}, & a^{-1}q^{-2}, & \ldots, & a^{-1}q^{1-D}, & a^{-1}q^{-D}. \end{array}$$

We abbreviate $\lambda_i := aq^i$ $(0 \leq i \leq D-1)$ and $\lambda_{-i} := a^{-1}q^{-i}$ $(1 \leq i \leq D)$. Let

$$\mathbf{y}_i = \omega_i E_i\hat{x} + \omega_i^{\perp} E_i v_0^{\perp}, \qquad \mathbf{y}_{-i} = \omega_{-i} E_i\hat{x} - \omega_i^{\perp} E_i v_0^{\perp} \qquad (1 \leq i \leq D-1),$$

where

$$\omega_i = \frac{a^2q^D(q^{i-D}-1)(q^{D+e+i}-q^{D+e}-q^{e+i}-q^i)-(q^D-q^i)(q^e+q^i)}{(q^D-1)(q^e+1)(a^2q^{2i}-1)},$$
$$\omega_{-i} = \frac{a^2q^D(q^i-1)(q^{D+e}+q^i)-(q^i-1)(q^e+1+q^i-q^D)}{(q^D-1)(q^e+1)(a^2q^{2i}-1)},$$
$$\omega_i^{\perp} = \frac{(a^2q^D-1)q^{i+1}}{(q^D-1)(q^e+1)(a^2q^{2i}-1)}.$$

We also let $\mathbf{y}_0 = E_0\hat{x}$ and $\mathbf{y}_{-D} = E_D\hat{x}$.

Proposition 7.3. *With the above notation, $\mathbf{y}_i$ is an eigenvector of $\mathbf{Y}$ for the eigenvalue λ_i for $-D \leq i \leq D-1$. Moreover, we have $\sum_{i=-D}^{D-1} \mathbf{y}_i = \hat{x}$.*

Lemma 7.4. *For $1 \leq i \leq D-1$, we have*

$$\|\mathbf{y}_i\|^2 = \omega_i^2 m_i + \omega_i^{\perp 2} m_i^{\perp} \|v_0^{\perp}\|^2, \qquad \|\mathbf{y}_{-i}\|^2 = \omega_{-i}^2 m_i + \omega_i^{\perp 2} m_i^{\perp} \|v_0^{\perp}\|^2,$$

where $\|v_0^{\perp}\|^2 = q^{e-1}(q^{D-1}-1)(q^D-1)$, and

$$m_i = \frac{(-1)^D (q^{-D};q)_i (1-a^2q^{2i})}{a^{2(i-D)} q^{i^2-Di-\frac{D(D+1)}{2}} (q;q)_i (a^2q^i;q)_{D+1}} \qquad (0 \leq i \leq D),$$

$$m_i^{\perp} = \frac{(-1)^{D-2} (q^{-D+2};q)_i (1-a^2q^{2i+2})}{a^{2(i-D+2)} q^{i^2-Di+2i-\frac{(D-2)(D-1)}{2}} (q;q)_i (a^2q^{i+2};q)_{D-1}} \qquad (0 \leq i \leq D-2).$$

Moreover, $\|\mathbf{y}_0\|^2 = m_0$ and $\|\mathbf{y}_{-D}\|^2 = m_D$.

Lemma 7.5. *For $f, g \in \mathcal{L}$, we have*

$$\langle f(\mathbf{Y}).\hat{x}, g(\mathbf{Y}).\hat{x} \rangle = \sum_{i=-D}^{D-1} f(\lambda_i)\overline{g(\lambda_i)} \|\mathbf{y}_i\|^2.$$

Define a Hermitian inner product $\langle \cdot, \cdot \rangle_{\mathcal{L}} : \mathcal{L} \times \mathcal{L} \to \mathbb{C}$ by

$$\langle f, g \rangle_{\mathcal{L}} = \sum_{i=-D}^{D-1} f(\lambda_i)\overline{g(\lambda_i)} \|\mathbf{y}_i\|^2 \qquad (f, g \in \mathcal{L}). \tag{7.1}$$

We are now ready to present the orthogonality relation for the non-symmetric dual q-Krawtchouk polynomials:

Theorem 7.6. *Let ℓ_i^+, ℓ_i^- be the Laurent polynomials from Definition 6.2. With reference to the inner product (7.1), we have*

$$\langle \ell_i^{\sigma}, \ell_j^{\tau} \rangle_{\mathcal{L}} = \delta_{\sigma,\tau} \delta_{i,j} \, |C_i^{\sigma}|$$

for $0 \leq i, j \leq D-1$ and $\sigma, \tau \in \{+, -\}$.

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